

# ON THE DECOMPOSITION OF THE SMALL DIAGONAL OF A K3 SURFACE

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**ABSTRACT.** We give a new proof of the theorem of Beauville and Voisin about the decomposition of the small diagonal of a K3 surface  $S$ . Our proof is explicit and works with the embedding of  $S$  in  $\mathbb{P}^g$ . It is different from the one used by Beauville and Voisin, which employed the existence of one-parameters families of elliptic curves.

The canonical zero cycle on a K3 surface  $S$  is defined in [1] as the rational equivalence class of any point lying on a rational curve  $C \subset S$ . The paper [1] shows that the intersection of any two divisors in  $S$  is proportional to the canonical cycle in  $\mathrm{CH}_0(S)$ . It is also shown that the second Chern class  $c_2(S)$  is proportional to this canonical zero cycle  $o$ . Both results can be obtained as consequences of the following theorem.

**Theorem 1.** ([1, Proposition 4.2]) *Let  $S$  be a K3 surface. In  $\mathrm{CH}_2(S^3)_{\mathbb{Q}}$  there is a decomposition*

$$(1) \quad \Delta_{123} = \Delta_{12} \times o_3 + \Delta_{23} \times o_1 + \Delta_{13} \times o_2 \\ - S \times o \times o - o \times S \times o - o \times o \times S,$$

where  $o$  is any point representing the canonical zero cycle,  $\Delta_{123}$  is a small diagonal in  $S^3$ , and the notation  $\Delta_{ij} \times o_k$  stands for  $\pi_{ij}^*(\Delta) \cdot \pi_k^*o$ .

The goal of this paper is to give another proof of Theorem 1 for a K3 surface  $S$  with  $\mathrm{Pic}(S) = \mathbb{Z}[L]$  with  $L^2 = 2g - 2$ . Our proof is very explicit using the embedding of  $S$  in  $\mathbb{P}^g$ . It is based on the study of the set of pairs of points  $(x, y)$  in  $S \times S$  such that two curves in the linear system  $|L|$  intersect exactly at these two points with given multiplicities. Specifically, we choose the multiplicities  $2g - 3$  and 1. In other words, we are studying the surface  $\Sigma$  parameterising complete intersections subschemes of  $S$  consisting in the union of two points, one of them with multiplicity  $2g - 3$ . We will prove that this is a surface and will establish two relations (6) and (7), from which we obtain the relation (1) up to some multiplicative factor  $\mu$ , which is non-zero if the surface  $\Sigma \subset S \times S$  dominates factors. The second part of the paper is then devoted to the proof that  $\mu \neq 0$ . In order to

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prove this non-vanishing we will interpret the surface  $\Sigma$  in a slightly different way: as  $\text{Pic}(S) = \mathbb{Z}[L]$ , the curves in  $|L|$  are irreducible hence the intersection of any two different curves in the linear system  $|L|$  is a zero-dimensional subscheme of  $S$  of length  $\deg(L)$ , so we have a morphism  $Gr(2, H^0(S, L)) \rightarrow S^{[2g-2]}$  and we let  $Gr$  denote the image. Using techniques from [2, 5, 6, 10] to work with cohomology groups of the Hilbert scheme  $S^{[2g-2]}$ , one can define the pieces  $E_M^*(Gr) \in \text{CH}(S^m)$  of the decomposition of the class of  $Gr$  in  $\text{CH}(S^{[2g-2]})$ , where  $M$  is a partition of  $\{1, \dots, 2g-2\}$  and  $m = |M|$ . The proof that  $\mu \neq 0$  involves the study of this class  $E_M^*(Gr)$  in the case where  $M$  is a partition into two integers.

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## 1. THE PROOF

1. **Surface  $\Sigma$ .** Let us recall that a K3 surface with a very ample linear system  $L$  of degree  $2g - 2 \geq 4$  generating  $\text{Pic}(S)$  can be embedded in  $\mathbb{P}^g$  and the intersection of  $S$  with any linear subspace  $\mathbb{P}^{g-2} \subset \mathbb{P}^g$  is a zero-cycle of degree  $2g - 2$  on  $S$ . The set of all  $P = \mathbb{P}^{g-2} \subset \mathbb{P}^g$  is the Grassmann variety  $Gr(g-1, g+1)$  and as already mentioned this provides a morphism

$$Gr(g-1, g+1) \rightarrow S^{[2g-2]},$$

which maps  $[P]$  to  $P \cap S$  for  $[P] \in Gr(g-1, g+1)$ . We denote by  $Gr$  the image of this map. We introduce the incidence scheme  $\Xi \subset S \times S^{[2g-2]}$ :

$$(2) \quad \begin{array}{ccc} \Xi & \xrightarrow{\pi_2} & S^{[2g-2]} \\ \downarrow \pi_1 & & \\ S & & \end{array}$$

Let us put  $\mathcal{L}^{[2g-2]} = (\pi_2)_* \pi_1^*(\mathcal{O}_S(L))$ . The image  $Gr$  is the locus in  $S^{[2g-2]}$ , where the rank of the map

$$H^0(S, H) \otimes \mathcal{O}_{S^{[2g-2]}} \rightarrow \mathcal{L}^{[2g-2]}$$

is  $g-1$  and it follows by [4] that its class is given by

$$(3) \quad c_{g-2}(\mathcal{L}^{[2g-2]})c_g(\mathcal{L}^{[2g-2]}) - c_{g-1}^2(\mathcal{L}^{[2g-2]}).$$

Let us also define a subset  $G'_0 \subset Gr(g-1, g+1)$  in the following way

$$G'_0 = \{[P] \in Gr(g-1, g+1) : P \cap S = (2g-3)p_1 + p_2 \text{ as a cycle}\}.$$

We also define the variety  $\Sigma_0 \subset S \times S$  as

$$\Sigma_0 := \{(p_1, p_2) \in S \times S : (2g-3)p_1 + p_2 = S \cap P \text{ for some } [P] \in Gr(g-1, g+1)\}.$$

If we construct the following diagram

$$(4) \quad \begin{array}{ccc} G'_0 \subset E_{2g-3,1} & \xrightarrow{q} & S^{[2g-2]} \\ \downarrow p & & \\ \Sigma_0 \subset S \times S & & \end{array}$$

where  $E_{2g-3,1}$  represents schemes of the form  $(2g-3)p_1 + p_2$  on  $S$ , we then have  $G'_0 = \pi_2^{-1}(Gr)$  and  $\Sigma_0 = \pi_1(\pi_2^{-1}(Gr))$ . Let us prove the following facts about geometry of  $\Sigma_0$ .

**Lemma 1.1.** *The following holds*

- (a)  $\Sigma_0$  is a non-empty surface, possibly reducible,
- (b) there is a component  $\Sigma \subset \Sigma_0$  such that  $\Sigma$  dominates both factors of  $S \times S$ .

*Proof.* Let us first proof that  $\dim \Sigma_0 \leq 2$ . Indeed, from the equality

$$p_1 + (2g-3)p_2 = L^2,$$

valid in  $\text{CH}^2(S)$  for any pair  $(p_1, p_2) \in \Sigma_0$ , and the theorem of Mumford [7], it follows that

$$(5) \quad \pi_1^* \sigma_S + (2g-3)\pi_2^* \sigma_S = 0$$

on  $\Sigma_0$ , where  $\sigma_S$  is a non-vanishing 2-form on  $S$ . Therefore  $\dim \Sigma_0 \leq 2$ .

Notice that equation (5) characterises K-correspondence in the terminology of [9]. This equation implies that for any irreducible component  $\Sigma$  of  $\Sigma_0$  the morphism  $\pi_1|_{\Sigma}$  is dominant if and only if the morphism  $\pi_2|_{\Sigma}$  is dominant. Indeed, these conditions are respectively equivalent to the generic non-vanishing of  $\pi_i^* \sigma_S$ . This argument also shows that (a) and (b) are implied by the fact that the first projection  $\pi_1|_{\Sigma_0}$  is dominant. In order to prove this last statement we observe that the cycle  $\Omega = p_* q^*(Gr)$  has for support the surface  $\Sigma_0$  although this cycle could be non-effective due to the fact that even if  $\Sigma_0$  has the right dimension the scheme  $G'_0 = q^{-1}(Gr)$  could be of a higher dimension leading to excess formulas in the computation of the cycle  $\Omega$ . Nevertheless we can argue that if  $\Omega$  can not be represented by a cycle supported on the union of divisors of the form  $D \times S$ , then one component of the support  $\text{supp } \Omega$  has to dominate  $S$  by the first projection, that is, one component of  $\Sigma_0$  dominates  $S$  by the first projection. The next section is devoted to the proof that the class  $\Omega$  can not be supported on the union of  $D \times S$ , see Proposition 2.1.  $\square$

**Remark 1.2.** Let us note that we expect  $\dim G'_0 = 2$ ,  $\Sigma_0$  is irreducible and the projection  $p : G'_0 \rightarrow \Sigma_0$  is a one-to-one correspondence. In this case we get  $[\Sigma_0] = \Omega$  in  $H^4(S \times S)$  (actually, we can consider the equality even in  $\text{CH}^2(S \times S)$ ). Unfortunately, the author does not know how to prove these facts and we avoid them in our proof by introducing below a surface  $G'$  as a substitute of  $G'_0$ .

Let  $\Sigma$  be a surface as in the last lemma and let  $G' \subset G'_0$  be any surface dominating  $\Sigma$  after the projection  $\pi_1 : G'_0 \rightarrow \Sigma_0$ . We can consider  $G'$  as a subvariety of  $Gr(g-1, g+1)$  and define  $\Pi_2 \subset \mathbb{P}^g \times \mathbb{P}^g$  and  $\Pi_3 \subset \mathbb{P}^g \times \mathbb{P}^g \times \mathbb{P}^g$  as the universal varieties:

$$\Pi_2 := \{(p_1, p_2) \in \mathbb{P}^g \times \mathbb{P}^g : x_1, x_2 \in P \text{ for } [P] \in G'\},$$

$$\Pi_3 := \{(p_1, p_2, p_3) \in \mathbb{P}^g \times \mathbb{P}^g \times \mathbb{P}^g : x_1, x_2, x_3 \in P \text{ for } [P] \in G'\}.$$

We have  $\dim \Pi_2 = \dim G' + 2(g-2) = 2g-2$  and  $\dim \Pi_3 = \dim G' + 3(g-2) = 3g-4$ . Clearly,  $\Sigma \subset \Pi_2 \cap S \times S$ .

**2. Main result.** A key observation for our proof is the following lemma.

**Lemma 1.3.** (a) *There is a decomposition in  $\text{CH}_2(S \times S)$  which, in fact, is an equality of effective cycles*

$$(6) \quad \Pi_2|_{S \times S} = \alpha \Delta + \beta (\Sigma + \Sigma^T),$$

where  $\Delta$  is the diagonal in  $S \times S$ .

(b) *There is a decomposition in  $\text{CH}_2(S \times S \times S)$*

$$(7) \quad \Pi_3|_{S \times S \times S} =$$

$$\gamma \Delta_{123} + \varepsilon (\delta_{12*}(\Sigma + \Sigma^T) + \delta_{23*}(\Sigma + \Sigma^T) + \delta_{31*}(\Sigma + \Sigma^T)),$$

where  $\Delta_{123}$  is the small diagonal in  $S^3$ , and  $\delta_{12}(x, y) = (x, x, y)$ ,  $\delta_{23}(x, y) = (y, x, x)$ ,  $\delta_{31}(x, y) = (x, y, x)$ .

*Proof.* The proof of (6) follows from the facts that  $\Pi_2|_{S \times S}$  is symmetric and supported on the union of the diagonal,  $\Sigma$  and  $\Sigma^T$ , and that  $\Sigma$  and  $\Sigma^T$  are chosen to be irreducible. The proof of (7) is similar.  $\square$

**Lemma 1.4.** *We have:*

(a) *the denominators of ratios  $\frac{\alpha}{\beta}$  and  $\frac{\gamma}{\varepsilon}$  are non-zero and both ratios are non-negative,*

(b) *the following relation holds*

$$(8) \quad \frac{\gamma}{\varepsilon} - \frac{3\alpha}{\beta} = - \left( \frac{\alpha}{\beta} + a + b \right),$$

where  $a$  and  $b$  are the degrees of the projections of  $\Sigma \subset S \times S$  to its factors.

*Proof.* (a) As numbers  $\alpha, \beta, \gamma, \varepsilon$  are non-negative and we need only to show that  $\beta, \varepsilon \neq 0$ . Since the diagonal  $\Delta$  can not be the restriction of a cycle from  $\mathbb{P}^g \times \mathbb{P}^g$ , we have  $\beta \neq 0$ , the proof of  $\varepsilon \neq 0$  is similar. (We use here the fact that  $S$  has some transcendental cohomology, so that the cohomology class of the diagonal of  $S$  does not vanish on a product  $U \times U$ , where  $U \subset S$  is dense Zariski open.)

(b) Projecting (7) to  $S \times S$  and taking cohomology classes, we easily conclude that  $\frac{\gamma}{\varepsilon} = \frac{2\alpha}{\beta} - a - b$ , which is equivalent to (8).  $\square$

*Proof of Theorem 1.* We chose a surface  $\Sigma$  as in Lemma 1.1. Due to [1, Proposition 2.6],  $\delta_{ij*}(\Pi_2|_{S \times S})$  can be represented by a sum of  $Z'|_{S^3}$  and  $o_k \times \Delta_{ij}$ . We also recall that  $\delta_{ij*}(\Delta) = \Delta_{123}$ . So, putting (6) and (7) together, we get a decomposition of the small diagonal:

$$(9) \quad \left( \frac{\gamma}{\varepsilon} - \frac{3\alpha}{\beta} \right) \Delta_{123} = \alpha_1 \Delta_{12} \times o_3 + \alpha_2 \Delta_{23} \times o_1 + \alpha_3 \Delta_{13} \times o_2 + Z|_{S^3},$$

where  $Z \subset \mathbb{P}^g \times \mathbb{P}^g \times \mathbb{P}^g$ .

Projecting to  $S \times S$  and taking the cohomology classes, we easily conclude that  $\alpha_1 = \alpha_2 = \alpha_3 = \gamma/\varepsilon - 3\alpha/\beta$ , and by previous lemma,  $\alpha_1 = -(\alpha/\beta + a + b)$ . Since the decomposition of the small diagonal holds in cohomology (due to [1] and more generally [8]), we can deal with the term  $Z|_{S^3}$  as follows: this term is a polynomial in  $L_1, L_2, L_3$ , where  $L_i := pr_i^* L$  and on the other hand it is cohomologous to

$$-\alpha_1(S \times o \times o + o \times S \times o + o \times o \times o \times S).$$

By [1] it is thus rationally equivalent to  $-\alpha_1(S \times o \times o + o \times S \times o + o \times o \times o \times S)$ . Since  $a, b > 0$  by choice of  $\Sigma$  and  $\alpha/\beta$  is non-negative, we can divide the equation by  $-\alpha/\beta - a - b$  to get the result. The theorem is proved.  $\square$

**Remark 1.5.** We would like to emphasise that this proof is very different from the one used by Beauville and Voisin, which uses the existence of one-parameters families of elliptic curves. It is much more along the lines of the method used by Voisin in the Calabi–Yau hypersurface case, and Fu in the Calabi–Yau complete intersection case (see [11, Theorem 3.1], [3]). To study the case of Calabi–Yau varieties, we need to replace  $G'$  by the set of lines intersecting the hypersurface in two points. In this case, the result [1, Proposition 2.6] used in our proof, becomes [11, Lemma 3.3].

**Remark 1.6.** As proved in [1], the decomposition of the small diagonal immediately gives the fact that  $c_2(S)$  is proportional to the canonical cycle  $o$ . From our proof we can easily get another more direct proof of this fact, using only (6). Indeed, let us intersect the decomposition (6) with  $\Delta$ . We get that  $\alpha c_2(S)$  is a combination of a canonical zero cycle (corresponding to the term  $\Pi_2|_{S \times S}$ ) and zero cycles supported on  $\Sigma \cap \Delta$  and  $\Sigma^T \cap \Delta$ . But clearly the points on  $\Sigma \cap \Delta$  are rationally equivalent to  $o$ . This proves the statement concerning  $c_2$ , once we prove that  $\alpha \neq 0$ , which can be derived from the following remark or from the proofs of Lemma 1.1 and of Proposition 2.1 in the next section.

**Remark 1.7.** Let us present a relation between  $\alpha/\beta$ ,  $a$ , and  $b$ . We see from the definition of  $\Sigma$  that for any  $(x, y) \in \Sigma$ , we have the equality

$$(2g - 3)x + y = L^2 \text{ in } \text{CH}_0(S).$$

It follows that we have for any  $x \in S$

$$\Sigma_*(x) = -a(2g - 3)x + C \text{ in } \text{CH}_0(S),$$

where  $C$  is a constant multiple of  $L^2$ . Similarly

$$\Sigma_*^T(x) = -\frac{b}{2g - 3}x + C' \text{ in } \text{CH}_0(S).$$

Applying (6) and the fact that  $\Pi_2$  is restricted from  $\mathbb{P}^g \times \mathbb{P}^g$ , we thus conclude that for any  $x \in S$

$$C'' = -a(2g - 3)x - \frac{b}{2g - 3}x + \frac{\alpha}{\beta}x \text{ in } \text{CH}_0(S),$$

where  $C''$  is a constant multiple of  $L^2$ . It follows that

$$\alpha/\beta = a(2g - 3) + \frac{b}{2g - 3}.$$

## 2. PROOF OF THE FACT THAT $\text{supp } \Omega$ DOMINATES FACTORS

The goal of this section is to prove the following lemma.

**Proposition 2.1.** *The class  $\Omega = E_{2g-3,1}^*(Gr)$  in  $H^*(S \times S)$  can not be represented by a cycle supported on the union of divisors of the form  $D_i \times S$  and  $S \times D_j$  and hence its support has non-trivial projections to factors of  $S \times S$*

To prove Proposition 2.1 we study  $H^*(S^{[2g-2]})$  and introduce the following notation. Let

$$M = (m_1, m_2, \dots, m_k)$$

be a partition of  $\{1, \dots, 2g-2\}$ . Such a partition determines a partial diagonal

$$S_M \cong S^k \subset S^{2g-2},$$

defined by the conditions

$$x = (x_1, \dots, x_{2g-2}) \in S_M \iff x_i = x_j \text{ if } i, j \in m_l, \text{ for some } l.$$

Consider the quotient map

$$q_M : S^k \cong S_M \rightarrow S^{(2g-2)},$$

and denote by  $E_M$  the following fibered product:

$$E_M := S_M \times_{S^{(2g-2)}} S^{[2g-2]} \subset S^k \times S^{[2g-2]}.$$

We view  $E_M$  as a correspondence between  $S^k$  and  $S^{[2g-2]}$  and we will denote by  $E_M^* : \text{CH}(S^{[2g-2]}) \rightarrow \text{CH}(S^m)$  the map

$$\alpha \rightarrow \pi_{1*}(\pi_2^*(\alpha) \cdot E_M).$$

The main point of the proofs is considering  $E_M^*(Gr)$  for the partition  $M = (\{1, \dots, 2g-3\}, \{2g-2\})$  and the intersection

$$\pi_{2*}(E_M) \cdot (c_{g-1}^2(\mathcal{L}^{[2g-2]}) - c_g(\mathcal{L}^{[2g-2]})c_{g-2}(\mathcal{L}^{[2g-2]}))$$

We now turn our attention to the cup product on Hilbert scheme  $S^{[2g-2]}$ .

1. **Cup product on  $S^{[n]}$ .** The paper [6] gives a description on the ring structure on  $H^*(S^{[n]})$ ; the following theorem holds (cf. [6, Theorem 3.2]):

**Theorem 2.2.** *Let  $S$  be a smooth projective surface with numerically trivial canonical class. Then there is a canonical isomorphism of graded rings*

$$(H^*(S; \mathbb{Q})[2])^{[n]} \rightarrow H^*(S^{[n]}; \mathbb{Q})[2n].$$

In the theorem above we define  $A^{[n]}$  as

$$A^{[n]} := (A\{S_n\})^{S_n}.$$

It is the subspace of invariants of the ring  $A\{S_n\}$ , which has the following grading by permutations in  $S_n$

$$A\{S_n\} := \bigoplus_{\pi \in S_n} A^{\otimes(\pi) \setminus [n]} \cdot \pi.$$

To describe  $c_i(\mathcal{O}^{[2g-2]})$  in these terms, let us introduce the following notation. If  $\sigma \in S_n$  is a permutation, then let  $c(\sigma)$  be the number

of cycles in  $\sigma$  and  $l(\sigma) = n - c(\sigma)$ . The number  $l(\sigma)$  is the minimal number of permutation needed to generate  $\sigma$ .

The statement [6, Proposition 4.3] (see also [5]) gives

$$(10) \quad c_i(\mathcal{O}^{[2g-2]}) = \epsilon_i, \text{ where } \epsilon_i := (-1)^i \sum_{l(\sigma)=i} \sigma \in H^*(S)^{[2g-2]}.$$

The class of  $E_M$  for  $M = (\{1, \dots, 2g-3\}, \{2g-2\})$  is proportional to the sum of all permutations which contains one cycle of length  $2g-3$ .

**2. Two lemmas and the proof.** Before we start the proof of Proposition 2.1, we would like to state two lemmas about transpositions. Let us enumerate all transpositions in  $S_{2g-3}$  by  $s_1, s_2, \dots, s_{(2g-3)(g-2)}$  in such a way that

$$l(s_1 \cdot \dots \cdot s_{2g-4}) = 2g-4$$

and define  $A(k)$  as the set of all permutations  $\sigma \in S_{2g-3}$  such that  $l(\sigma s_i) > l(\sigma)$  for any  $i \leq k$ . We note that for  $\sigma \in A(k)$  one has  $\sigma \in A(k+1)$  or  $\sigma = \sigma' s_{k+1}$  for some  $\sigma' \in A(k+1)$ .

Let us define set of pairs  $\mathcal{F}_k(i, j, \tau)$ :

$$\{(\sigma_1, \sigma_2) \in A_k \times A_k : l(\sigma_1) = i, l(\sigma_2) = j, l(\sigma_1 \sigma_2 \tau) = i + j + l(\tau)\}.$$

And let  $F_k(i, j, \tau)$  be the number of elements in  $\mathcal{F}_k(i, j, \tau)$ .

**Lemma 2.3.** *If  $i+1 < j$ , one has*

$$(11) \quad F_k(i, j, \tau) \leq F_k(i+1, j-1, \tau).$$

*and the inequality is strict in the case  $i = g-3$ ,  $j = g-1$ ,  $k = 0$ ,  $\tau = id$ .*

*Proof.* The set  $\mathcal{F}_k(i, j, \tau)$  can be divided in four subsets:

- (a) pairs  $(\sigma_1, \sigma_2)$  such that  $\sigma_1, \sigma_2 \in A(k+1)$ . This subset coincides with  $\mathcal{F}_{k+1}(i, j, \tau)$ .
- (b) pairs  $(\sigma_1, \sigma_2)$  such that  $\sigma_1 \in A(k+1)$  and  $\sigma_2 \notin A(k+1)$ , so  $\sigma_2 = \sigma'_2 s_{k+1}$  with  $l(\sigma'_2) = j-1$ . This subset is in bijection with pairs  $(\sigma_1, \sigma'_2)$  of  $\mathcal{F}_{k+1}(i, j-1, s_{k+1}\tau)$ .
- (c) pairs  $(\sigma_1, \sigma_2)$  such that  $\sigma_1 \notin A(k+1)$  and  $\sigma_2 \in A(k+1)$ , so  $\sigma_1 = \sigma'_1 s_{k+1}$  with  $l(\sigma'_1) = i-1$ . This subset is in bijection with pairs  $(\sigma'_1, s_{k+1}\sigma_2 s_{k+1})$  of  $\mathcal{F}_{k+1}(i-1, j, s_{k+1}\tau)$ .
- (d) pairs  $(\sigma_1, \sigma_2)$  such that  $\sigma_1, \sigma_2 \notin A(k+1)$ , so  $\sigma_1 = \sigma'_1 s_{k+1}$  and  $\sigma_2 = s_{k+1}\sigma'_2$ , hence  $l(\sigma_1 \sigma_2 \tau) \leq l(\sigma_1) \cdot l(\sigma_2) \cdot l(\tau) - 2$ . This subset is empty.



So we have

$$(12) \quad F_k(i, j, \tau) = F_{k+1}(i, j, \tau) + F_{k+1}(i-1, j, s_{k+1}\tau) + F_{k+1}(i, j-1, s_{k+1}\tau).$$

The proof easily follows by induction if we prove that

$$F_k(0, j, \tau) \leq F_k(1, j-1, \tau)$$

for all  $j, k$  and all  $\tau$ . Inequality follows from the fact that any permutation  $\sigma_2$ , where  $(id, \sigma_2) \in \mathcal{F}_k(0, j, \tau)$ , has (more than one) decomposition  $\sigma'_1 \sigma'_2$ , where  $(\sigma'_1, \sigma'_2) \in \mathcal{F}_k(1, j-1, \tau)$ . Different  $\sigma_2$  has different decompositions. We note that the inequality is strict if the lefthand side is positive, i.e., there exist at least one  $\sigma_2$ . We can consider

$$(\sigma_1, \sigma_2) = (s_1 \dots s_{g-2}, s_{g-1} \dots s_{2g-4})$$

which provide with a non-emptiness of  $F_{g-2}(0, g-2, \sigma_1^{-1})$ .  $\square$

Let  $\mathcal{G}(i, j)$  be the set of all pairs  $(\sigma_1, \sigma_2) \in S_{2g-2} \times S_{2g-2}$  such that

$$l(\sigma_1) + l(\sigma_2) - 2 = l(\sigma_1 \sigma_2)$$

and with the composition  $\sigma_1 \sigma_2$  preserves the point  $2g-2$ :  $(\sigma_1 \sigma_2)(2g-2) = 2g-2$ . Let  $G(i, j)$  be the number of elements in  $\mathcal{G}(i, j)$ .

**Lemma 2.4.** *One has*

$$G(g-1, g-1) > G(g-2, g).$$

*Proof.* Let us define a map of sets  $f : S_n \rightarrow S_{n-1}$  by the following way: if  $\sigma$  has a cycle  $(\dots, i, n, j, \dots)$  we replace it by a cycles  $(\dots, i, j, \dots)$ . Clearly, if  $\sigma(n) \neq n$  than  $l(\sigma') = l(\sigma) - 1$ .

Now it is easy to see that  $f(\mathcal{G}(i, j))$  is  $(2g-3)$ -fold covering of  $\mathcal{F}_0(i-1, j-1, id)$ , where  $\mathcal{F}_0(i-1, j-1, id)$  was defined before Lemma 2.3. Due to Lemma 2.3, we get the result.  $\square$

*Proof of Proposition 2.1.* We recall that class of  $Gr$  is given by (3). Using Grothendieck-Riemann-Roch theorem it is easy to see that classes  $c_k(\mathcal{L}^{[2g-2]})$  are polynomials in  $L_i$  and in other classes. We will be interesting in the coefficient of the diagonal in the decomposition of  $E_{2g-3,1}^*(Gr)$ , so we pay attention only to the part without  $L_i$ . Now we can put formally  $L = 0$  and we consider the class

$$c_{g-1}^2(\mathcal{O}^{[2g-2]}) - c_{g-2}(\mathcal{O}^{[2g-2]})c_g(\mathcal{O}^{[2g-2]}).$$

The classes  $c_i(\mathcal{O}^{[2g-2]})$  are given by (10) and we need to understand the sum

$$\sum_{M, l(\sigma_1)=g-1, l(\sigma_2)=g-1} \sigma_M \sigma_1 \sigma_2 - \sum_{M, l(\sigma_1)=g-2, l(\sigma_2)=g} \sigma_M \sigma_1 \sigma_2$$

Every summand correspond to a class of  $c_2(S) \times S$  (or  $S \times c_2(S)$ ) or  $\Delta$  in  $S \times S$ . To distinguish the classes of diagonals, we need to require the factor  $e^{g(\sigma_M, \sigma_1, \sigma_2)}$ , which appears in triple intersection, is equal to one (cf. [6, Proof of Lemma 2.13]). It implies that three following conditions hold

- (a)  $l(\sigma_1 \sigma_2) = l(\sigma_M)$ , in particular,  $l(\sigma_1 \sigma_2) = l(\sigma_1) + l(\sigma_2) - 2$ ,
- (b) there is an element  $i \in \{1, \dots, 2g-2\}$  such that  $(\sigma_1 \sigma_2)(i) = i$ ,
- (c) there is no element  $i \in \{1, \dots, 2g-2\}$  such that  $i = \sigma_1(i) = \sigma_2(i)$ .

Actually, the first two condition follows from the requirement  $\sigma_M \sigma_1 \sigma_2 = id$ .

The pairs  $(\sigma_1, \sigma_2)$  with  $l(\sigma_1) = i$ ,  $l(\sigma_2) = j$  satisfying these requirements are precisely the set  $\mathcal{G}(i, j)$  defined previously. Since  $G(g-1, g-1) > G(g-2, g)$  by Lemma 2.4, we get that the class  $\Omega$  contains diagonal. Therefore  $\Omega$  viewed as a self-correspondence of  $S$  does not act trivially on  $H^{2,0}(S)$  and can not be supported on divisors of the form  $D_i \times S$  and  $S \times D_j$ .  $\square$

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